Some Remarks on Balayage Measures

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1. Let E be a locally convex Hausdorff topological real linear space and K be a compact convex subset of E. Denote by $\mathscr{C}$ (respectively $\mathscr{S}$) the collection of real-valued continuous (respectively continuous concave) functions defined on K and by $\mathscr{M}$ the collection of Radon measures on K.

We define an order on $\mathscr{M}_+=\{\mu\in \mathscr{M};\ \mu$ is positive$\}$, introduced by G. Choquet[1].

Let λ, $\mu\in \mathscr{M}_+$. We say that μ is a balayage of λ if $\int fd\lambda\geqq\int fd\mu$ for all $f\in \mathscr{S}$, which we denote by $\lambda\prec\mu$, furthermore we say that a positive measure λ is maximal if and only if $\lambda\prec\mu$ implies $\lambda=\mu$ for any $\mu\in\mathscr{M}_+$.

Following G. Mokobodzki[4], we define for $f\in\mathscr{C}$,

$\hat{f}=\inf\ \{g\in\mathscr{S};\ g\geqq f$ on $K\}$. Becauce this function is bounded and upper semi-continuous, for every measure $\lambda\in\mathscr{M}_+$, we can set

$P_\lambda(f)=\int\hat{f}d\lambda=\inf\left\{\int gd\lambda;\ g\in\mathscr{S}\text{ and } g\geqq f\text{ on } K\right\}$, where the second equality follows from the fact that the set $\{g\in\mathscr{S},\ g\geqq f$ on $K\}$ is filtering to the left. In this situation, P. A. Meyer[3] proved that $P_\lambda(f)=\sup\left\{\int fd\mu;\ \mu\in\mathscr{M}_+\text{ and }\lambda\prec\mu\right\}$.

Here, we recall the theory of linear programming problems in paired spaces established by K. S. Kretschmer[2].

Let X and Y be real linear spaces paired under $((\ ,\ ))_1$ and Z and W be real linear spaces paired under $((\ ,\ ))_2$. The weak topology on X is denoted by $w(X, Y)$. Let P be a $w(X, Y)$-closed convex cone, Q be a $w(Z, W)$-closed convex cone, A be a linear transformation from X into Z which is $w(X, Y)-w(Z, W)$ continuous, y_0 be an element of Y and z_0 be an element of Z. In this situation, Kretschmer proved:

Theorem 1. If there exists an $x'\in P$ such that $Ax'-z_0\in Q^0$ and $\inf\ \{((x, y_0))_1\ ;\ x\in P$ and $Ax-z_0\in Q\}$ is finite, then there exists a $w_0\in Q^+$ such that $y_0-A^*w_0\in P^+$ and

$\inf\ \{((x, y_0))_1\ ;\ x\in P$ and $Ax-z_0\in Q\}$

$=\max\ \{((z_0, w))_2\ ;\ w\in Q^+$ and $y_0-A^*w\in P^+\}=((z_0, w_0))_2$, where $P^+=\{y\in Y;\ ((x, y))_1\geqq 0$ for all $x\in P\}$,

$Q^+=\{w\in W;\ ((z, w))_2\geqq 0$ for all $z\in Q\}$, A^* is the dual transformation of A, i. e. $((x, A^*w))_1=((Ax, w))_2$ for all $x\in X$ and $w\in W$ and Q^0 is the non-empty interiors of Q in the sense of the Mackey topology on Z.

Now, we apply Theorem 1 to the above theory.

Set $X=Z=\mathscr{C}$ with the uniform convergence topology, $Y=W=\mathscr{M}$, $P=\mathscr{S}$, $Q=\mathscr{C}_+=\{f\in\mathscr{C};\ f\geqq 0$ on $K\}$, $A=I$, which is the identity mapping of $\mathscr{C}$. Then $P^+=\{\lambda\in\mathscr{M};$

$\int fd\lambda \geqq 0$ for all $f \in \mathscr{S}\}$, and $Q^+ = \mathscr{M}_+$. Hence λ, $\mu \in \mathscr{M}_+$ and $\lambda - \mu \in P^+$ imply $\lambda \prec \mu$. Clearly I^* is the identity mapping of $\mathscr{M}$. In this case, Q has interiors Q^0 and for each $f \in \mathscr{C}$ there exists $g \in P$ such that $g - f \in Q^0$, furthermore $P_\lambda(f)$ is finite for $\lambda \in \mathscr{M}_+$. Hence we have:

Theorem 2. For $f \in \mathscr{C}$, $\lambda \in \mathscr{M}_+$ there exists a $\mu_0 \in \mathscr{M}_+$ such that $\lambda \prec \mu_0$ and

$$P_\lambda(f) = \max\left\{\int fd\mu \,;\, \mu \in \mathscr{M}_+ \text{ and } \lambda \prec \mu\right\} = \int fd\mu_0.$$

In general, this μ_0 depends on f and λ. If μ_0 does not depend on f, then λ is maximal. In fact, then for any μ such that $\lambda \prec \mu$ and for any $\phi \in \mathscr{C}$, $\int \phi d\mu \leqq \int \phi d\mu_0$, hence $\mu = \mu_0$. On the other hand, $\int \phi d\mu_0 = \int \hat{\phi} d\lambda \geqq \int \phi d\lambda$ for any $\phi \in \mathscr{C}$, hence $\mu_0 = \lambda$, i.e. $\mu = \lambda$. Inversely if λ is maximal, then $\mu \in \mathscr{M}_+$ sucsh that $\lambda \prec \mu$ is only λ.

Theorem 3. λ is maximal if and only if μ_0 does not depend on f.

Remark. If f is bounded upper semi-continuous, then

$$P_\lambda(f) = \max\left\{\int fd\mu \,;\, \mu \in \mathscr{M}_+ \text{ and } \lambda \prec \mu\right\}.$$

Indeed, let X and Y be the linear spaces consisting of bounded Borel functions on K with uniform convergence topology, P be $\mathscr{S}$ and Q be the collection of non-negative Borel functions. Then, the dual space of X contains $\mathscr{M}$ by the Riesz representation theorem, therefore there exists a continuous linear functional Φ_0 on X such that $\int gd\lambda \geqq \Phi_0(g)$ for all $g \in P$, $\Phi_0(\phi) \geqq 0$ for $\phi \in Q$ and

$P_\lambda(f) = \max\{\Phi(f)\,;\, \Phi$ is positive continuous linear functional on X and $\int gd\lambda \geqq \Phi(g)$ for all $g \in P\} = \Phi_0(f)$.

Let μ_0 be the positive Radon measure corresponding to Φ_0, i.e.

$$\Phi_0(\phi) = \int \phi d\mu_0 \text{ for any } \phi \in \mathscr{C}.$$

Since $\Phi_0(g) = \int gd\mu_0$ for any $g \in \mathscr{S}$, we have $\lambda \prec \mu_0$. Furthermore $\Phi_0(f) \leqq \int fd\mu_0$ as f is upper semi-continuous. On the other hand,

$\Phi_0(f) \geqq \int fd\mu$ for any $\mu \in \mathscr{M}_+$ such that $\lambda \prec \mu$, in particular for μ_0.

Consequently

$$\Phi_0(f) = \int fd\mu_0.$$

2. Let K e an arbitrary compact space, and $\mathscr{C}$, $\mathscr{M}$, $\mathscr{C}_+$ and $\mathscr{M}_+$ be as same as defined in the section 1. Let $\mathscr{S}$ be a non-empty closed convex cone in $\mathscr{C}$ such that $f \in \mathscr{S}$ and $f' \in \mathscr{S}$ imply $\min(f, f') \in \mathscr{S}$ and $\mathscr{S}$ contains the constant 1. We denote by $\prec$ the partial order on $\mathscr{M}_+$ defined by

$$(\lambda \prec \mu) \iff \left(\int fd\lambda \geqq \int fd\mu \text{ for every } f \in \mathscr{S}\right).$$

We say that a subset L of K is a Shilov set if the relations:

$f \in \mathscr{C}$ and $\inf\{f(x)\,;\, x \in L\} \geqq -1$

imply the inequality:

$\inf\{f(x)\,;\, x \in K\} \geqq -1$.

For a Shilov set L and $f \in \mathscr{S}$, we set

$\hat{f}_L = \inf \{g\,;\, g \in \mathscr{S} \text{ and } g \geqq f \text{ on } L\}$.

Since L is a Shilov set, $\hat{f}_L$ is bounded and upper semi-continuous, so we can define for $\lambda \in \mathscr{M}_+$,

$$P_{L,\lambda}(f) = \int \hat{f}_L d\lambda.$$

By the assumption on $\mathscr{S}$,

$$P_{L,\lambda}(f) = \inf\left\{\int g d\lambda\,;\, g \in \mathscr{S} \text{ and } g \geqq f \text{ on } L\right\}.$$

Furthermore, we have for any compact set L,

$$P_{L,\lambda}(f) = \sup\left\{\int f d\mu\,;\, \mu \in \mathscr{M}_+,\ \lambda \prec \mu \text{ and } \mu \text{ is carried by } \mathring{L}\right\}.$$

Again, we apply the theory of linear programming problems to the above.

Let $X = Z = \mathscr{C}$, $Y = W = \mathscr{M}$, $P = \mathscr{S}$ and $Q = \{f \in \mathscr{C};\ f \geqq 0 \text{ on } L\}$.

Then $\mu \in Q^+$ is a positive Radon measure carried by L if L is compact.

In fact, for every $f \in \mathscr{C}_+$, $\int f d\mu \geqq 0$ as $f \in Q$, i. e. $\mu \in \mathscr{M}_+$.

Further, if $f \geqq 0$ and $f = 0$ on L,

$$\int f d\mu = 0,$$

i. e. μ is carried by L.

We have:

Theorem 4. For $\lambda \in \mathscr{M}_+$, $f \in \mathscr{S}$ and a compact Shilov subset L of K, there exists $\mu_0 \in \mathscr{M}_+$ such that $\lambda \prec \mu_0$, μ_0 is carried by L and

$$P_{L,\lambda}(f) = \max\left\{\int f d\mu\,;\, \mu \in \mathscr{M}_+,\ \lambda \prec \mu \text{ and } \mu \text{ is carried by } L\right\} = \int f d\mu_0.$$

Remark. By the same reasoning as in the remark in the section 1, Theorem 4 holds if f is a bounded upper semi-continuous function.

References

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