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Article

The Number of Consecutive Heads in a Run

Hideo HIROSE*

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Abstract

How many consecutive heads do we observe in a run of coin tossing of length n? Although the problem seems to be easy to answer, this would be actually a little bit tough when we try to prove it straightforwardly. The expected number of consecutive heads in a run is $\frac{3n-2}{8}$ $(n \geq 2)$ using the recursive formula.

However, if we define a solitary head coin such that a head coin is isolated by neighboring tail coin(s) in a run, the problem of how many solitary heads we observe in a run can be solved easily. The expected number of solitary heads in a run is $\frac{n+2}{8}$ $(n \geq 2)$. Since the problem of solitary head coin becomes a dual problem of the above, the consequence of the problem of the consecutive heads is derived easily by considering the probability of a solitary coin appearance.

Key Words: coin tossing, run, consecutive heads, solitary head coin, dual problem

1 Introduction

There have been number of good discussions in coin tossing. Feller (1968) is most referred to, and very intriguing subjects are also discussed; e.g., Mood (1940), Bloom (1996), Finch (2003), Havil (2003), Gordon, Schilling, Waterman (1986), Philippou (1986), Schilling (1990), Schuster (1994) are among them. Some are in ideally fair coins, and others are in actual coin, e.g., Keller (1986), Ford (1983). Here we want to deal with the number of consecutive heads in a run.

Imagine that we are doing a run of coin tossing of length n. When $n = 3$, for example, the head (H) and tail (T) patterns are, HHH, HHT, HTH,..., TTT; the number of all possible patterns is $2^3 = 8$. Among these eight patters, we find three consecutive heads patterns; two consecutive heads cases are HHT and THH, and three consecutive heads case is HHH. The number of consecutive heads is counted to be two, two or three to each case. When $n = 5$, there are no consecutive heads in a run of THTTH or HTHTH; when a run is HHTHH, the number of consecutive heads becomes four; when THHHT, it is three.

In this paper, we consider a problem. How many consecutive heads can we expect to observe in a run of coin tossing of length n? To deal with the problem, we append one point to a head coin of consecutive heads and zero point otherwise. We demonstrate the case of $n = 3$ to grasp the problem. When $n = 3$, we get 3 point from HHH and 2 point each from THH or HHT, as shown in Figure 1; thus, the expected point in a run is $\frac{3+2+2}{8} = \frac{7}{8}$.

2 Expected Number of Consecutive Heads in a Run

We define a_i such that

$a_i = 1$ $\quad i$th flipped coin is head,

* Dept. of Architectural Design & Center for Data Science Research

$a_i = 0$ ith flipped coin is tail,

and b_i such that

$b_i = 1$ ith flipped coin is one of the consecutive heads,

$b_i = 0$ ith flipped coin is a solitary head or tail.

We, then, can define the point $t_n(a_1, a_2, \cdots, a_n)$ in a run by

$$t_n(a_1, a_2, \cdots, a_n) = \sum_{i=1}^{n} b_i.$$

By summing up t_n for all possible runs, we define the total possible point of

$$f(n) = \sum_{a_1, a_2, \cdots, a_n} t_n(a_1, a_2, \cdots, a_n).$$

As an example, we show a_i, t_n, and $f(n)$ for $n = 2,3$ in Figure 1. Once we can obtain $f(n)$, the expected point, $E[t_n]$, becomes to

$$E[t_n] = \frac{f(n)}{2^n}$$

a_1, a_2	t_2
1, 1	2
0, 1	0
1, 0	0
0, 0	0

$f(2) = 2$

	a_1, a_2, a_3		t_3
(i)	1, 1, 1	(ii)	3
	0, 1, 1		2
	1, 0, 1		0
	0, 0, 1	(iii)	0
(i)	1, 1, 0		2
	0, 1, 0		0
	1, 0, 0		0
	0, 0, 0		0

$f(3) = 7$

Figure 1 An example of a_i, t_i, and $f(n)$ for $n = 2,3$.

To $f(n)$, we consider a recursive formula. The total point consists of the following three in n coin tossing:

1) Whatever the value of a_n is, $f(n-1)$ expressed by $a_1, a_2, \cdots, a_{n-1}$ is taken into account of; i.e., $2f(n-1)$ is counted in $f(n)$. This is shown in (i) in Figure 1.

2) If $a_{n-1} = 1$, and $a_n = 1$, then $b_n = 1$. Thus, 2^{n-2} point are counted in $f(n)$, since we have 2^{n-2} possible cases for $a_1, a_2, \cdots, a_{n-2}$. This is shown in (ii) in Figure 1.

3) If $a_{n-2} = 0$, $a_{n-1} = 1$, and $a_n = 1$, then $b_{n-1} = 1$. When we deal with $n-1$ coin tossing, $b_{n-1} = 0$ if $a_{n-2} = 0$, $a_{n-1} = 1$. Thus, b_{n-1} changes its value from 0 to 1. According to this, 2^{n-3} point are counted in $f(n)$ because we have 2^{n-3} possible cases for $a_1, a_2, \cdots, a_{n-3}$. This is shown in (iii) in Figure 1.

Therefore, we have the recursive formula as

$$f(n) = 2f(n-1) + 2^{n-2} + 2^{n-3}, \quad (n \geq 3).$$

This formula can be solved as follows:

$$\begin{aligned}
f(n) &= 2f(n-1) + 3 \cdot 2^{n-3} \\
&= 2\left(2f(n-2) + 3 \cdot 2^{n-4}\right) + 3 \cdot 2^{n-3} \\
&= 2^2 f(n-2) + 3 \cdot 2^{n-3} + 3 \cdot 2^{n-3} \\
&= 2^2 f(n-2) + 3 \cdot 2 \cdot 2^{n-3} \\
&= 2^2 \left(2f(n-3) + 3 \cdot 2^{n-5}\right) + 3 \cdot 2 \cdot 2^{n-3} \\
&= 2^3 f(n-3) + 3 \cdot 2^{n-3} + 3 \cdot 2 \cdot 2^{n-3} \\
&= 2^3 f(n-3) + 3 \cdot 3 \cdot 2^{n-3} \\
&\cdots \\
&= 2^{n-4}\left(2f(3) + 3 \cdot 2^1\right) + 3(n-4)2^{n-3} \\
&= 2^{n-3} f(3) + 3(n-3)2^{n-3} \\
&= 2^{n-3}\left(f(3) + 3(n-3)\right) \\
&= 2^{n-3}\left(3(n-2)\right), \ (n \geq 3).
\end{aligned}$$

Therefore, the expected point, $E[t_n]$, which is equivalent to the expected number of consecutive heads, C_n, becomes

$$E[t_n] = \frac{2^{n-3}(3n-2)}{2^n} = \frac{3n-2}{8}, \quad (n \geq 2),$$

in a run, because this formula holds also when $n = 2$.

3 Probability That a Coin is a Solitary Head Coin

When $a_1 = 1$ and $a_2 = 0$, then the very first flipped coin is the solitary head coin. Whatever values the other a_i have, the probability that the first flipped coin is a solitary head coin is $\frac{1}{4}$ because the probability that $a_1 = 1$ and $a_2 = 0$, and $a_i = 0,1 (3 \leq i \leq n)$ is $\frac{2^{n-2}}{2^n}$. Let's consider here that we append point one only to the solitary head coin. Then, the expected point from this coin is $1 \times \frac{1}{4} = \frac{1}{4}$. This is also true for the very last flipped coin.

For the second flipped coin, it becomes a solitary head coin if $a_1 = 0$, $a_2 = 1$, and $a_3 = 0$, whatever values the other coins have, where $4 \leq i \leq n$. Then, the expected point from this coin is $1 \times \frac{1}{8} = \frac{1}{8}$, and this is also true for $a_3, \cdots, a_{n-2}$.

Therefore, the total expected point for the solitary heads, which is equivalent to the expected number of solitary heads, S_n, in a run becomes,

$$S_n = 2 \times \frac{1}{4} + (n-2) \times \frac{1}{8} = \frac{n+2}{8}.$$

Considering that the problem of how many solitary heads we observe in a run becomes a dual problem for the original consecutive heads observation problem, the expected number of consecutive heads in a run, C_n, is

$$C_n = \frac{n}{2} - \frac{n+2}{8} = \frac{3n-2}{8}, \ (n \geq 2).$$

4 Concluding Remarks

How many consecutive heads we observe in a run of coin tossing of length n, which seems to be a little bit tough to solve, can be solved easily by considering the dual problem of counting solitary head coins such that a head coin is isolated by neighboring tail coin(s) in a run. Since the expected number of solitary heads in a run is $\frac{n+2}{8}\ (n \geq 2)$, the expected number of consecutive heads in a run is $\frac{3n-2}{8}\ (n \geq 2)$.

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